STOKES VECTOR REPRESENTATION OF THE SIX-PORT NETWORK ANALYZER: CALIBRATION AND MEASUREMENT

Tokuo Oishi* and Walter K. Kahn **

*Computer & Data Systems Division, World Systems, COMSAT, Washington, D.C., U.S.A.

**Department of Electrical Engineering & Computer Science George Washington University, Washington, D.C., U.S.A.

SUMMARY

Calibration of six-port network analyzer is shown to be equivalent to the determination of four Stokes vectors. Analyzer performance can then be visualized in terms of the Stokes vector representing the unknown, relative to those representing the analyzer.

INTRODUCTION

Since Engen and Hoer introduced the six-port analyzer substantial literature has developed dealing with different six-port circuit configurations, calibration of the six-port, and measurement error analysis [1-6]. This paper formulates the connection between power meter indications and the value of the unknown reflection coefficient (or other complex ratio) in terms of Stokes vectors. In particular it is shown that each power meter indication can be expressed in terms of the scalar product of one of four Stokes vectors characteristic of six-port and a Stokes vector describing the unknown. The connection between Stokes vectors, the Poincare sphere and reflection coefficient provides an attractive geometrical interpretation which can provide valuable insight into the six-port calibration and measurement processes.

THEORY

A six-port analysis, with ports designated as shown in Fig. 1, will be represented by its scattering matrix $S = [S_{mn}]$. The scattering matrix is partitioned, separating out the input port (1) and the measurement port (2) from the power meter ports (3), (4), (5) and (6).

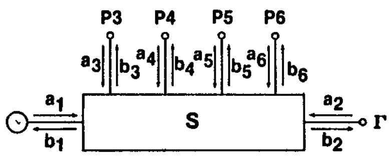

FIGURE 1. SIX-PORT MEASUREMENT SYSTEM

$$\underline{b} = \begin{bmatrix} \underline{b}_\alpha \\ \underline{b}_\beta \end{bmatrix} = \begin{bmatrix} S_{\alpha\alpha} & S_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{bmatrix} \begin{bmatrix} \underline{a}_\alpha \\ \underline{a}_\beta \end{bmatrix} = S\,\underline{a} \qquad (1)$$

where

$$\underline{a}_\alpha = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad \underline{a}_\beta = \begin{bmatrix} a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \text{, etc.}$$

The power detectors may be mismatched with reflection coefficients Q_n. Writing the diagonal matrix $Q = [Q_n \, \delta_{nm}]$, we have

$$\underline{a}_\beta = Q\,\underline{b}_\beta \qquad (2)$$

This relation may be substituted in (1) to obtain

$$\underline{b}_\beta = (1 - S_{\beta\beta}Q)^{-1} S_{\beta\alpha}\underline{a}_\alpha \qquad (3a)$$

$$= \underline{A}_\beta \, a_1 + \underline{B}_\beta \, a_2 \qquad (3b)$$

The diagonal elements of the matrix $\underline{b}_\beta \, \underline{b}_\beta^\dagger$ are the (average) power quantities $\hat{P}_n = b_n b_n^*$, incident on the detectors, n = 3, 4, 5, 6. In matrix form

$$\hat{\underline{P}} = \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \\ \hat{P}_4 \end{bmatrix} = \begin{bmatrix} |A_3|^2 & A_3B_3^* & B_3A_3^* & |B_3|^2 \\ |A_4|^2 & A_4B_4^* & B_4B_4^* & |B_4|^2 \\ |A_5|^2 & A_5B_5^* & B_5B_5^* & |B_5|^2 \\ |A_6|^2 & A_6B_6^* & B_6B_6^* & |B_6|^2 \end{bmatrix} \begin{bmatrix} |a_1|^2 \\ a_1a_2^* \\ a_2a_2^* \\ |a_2|^2 \end{bmatrix}$$

$$= C\,\hat{\underline{a}} \qquad (4)$$

Notice that the rows of C have the same structure as column vector $\hat{\underline{a}}$.

We now introduce the Stokes vector $\underline{a}$ via the linear transformation T.

$$\hat{\underline{a}} = \begin{bmatrix} |a_1|^2 \\ a_1a_2^* \\ a_2a_1^* \\ |a_2|^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & j & 0 \\ 0 & 1 & -j & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} |a_1|^2 + |a_2|^2 \\ 2\mathrm{Re}\; a_1a_2^* \\ 2\mathrm{Im}\; a_1a_2^* \\ |a_2|^2 - |a_1|^2 \end{bmatrix} = T\,\underline{a} \qquad (5)$$

so that

$$\hat{\underline{P}} = C\,T\,\underline{a} \qquad (6)$$

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The rows of the square matrix 2CT, $\underline{\mathcal{S}}_n$

$$2CT = \begin{bmatrix} \mathcal{S}_{30} & \mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33} \\ \mathcal{S}_{40} & \mathcal{S}_{41} & \mathcal{S}_{42} & \mathcal{S}_{43} \\ \mathcal{S}_{50} & \mathcal{S}_{51} & \mathcal{S}_{52} & \mathcal{S}_{53} \\ \mathcal{S}_{60} & \mathcal{S}_{61} & \mathcal{S}_{62} & \mathcal{S}_{63} \end{bmatrix} = \begin{bmatrix} \underline{\mathcal{S}}_3 \\ \underline{\mathcal{S}}_4 \\ \underline{\mathcal{S}}_5 \\ \underline{\mathcal{S}}_6 \end{bmatrix} \quad (7)$$

also have the form of Stokes vectors. For example,

$$\mathcal{S}_{30} = |B_3|^2 + |A_3|^2, \quad \mathcal{S}_{31} = 2\mathrm{Re}\ B_3A_3^* \quad (8)$$

$$\mathcal{S}_{32} = 2\mathrm{Im}\ B_3A_3^* \quad , \quad \mathcal{S}_{33} = |B_3|^2 - |A_3|^2; \text{ etc.}$$

Writing the transpose of $\underline{a}$,

$$\tilde{\underline{a}} = \begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} \quad (9)$$

we can express the incident power quantities as the product of Stokes vectors in accordance with (6).

It will be recalled that the 4-dimensional Stokes vectors may be given a geometrical interpretation in 3-dimensional space (Poincare sphere) [7,11]. The ordinary 3-dimensional vector with cartesian components (s_1, s_2, s_3)is written.

$$\vec{a} = s_1\ \hat{x} + s_2\ \hat{y} + s_3\ \hat{z}\ , \quad (10)$$

$$|\vec{a}|^2 = s_1^2 + s_2^2 + s_3^2 = s_0^2 \quad (11)$$

Therefore, the product of Stokes vectors (4) may be visualized in terms of the scalar product of ordinary vectors. Thus

$$2\hat{P}_n = \underline{\mathcal{S}}\ \underline{a} = \mathcal{S}_{n0}\ s_0 + \vec{\mathcal{S}}_n \cdot \vec{a} \quad (12a)$$

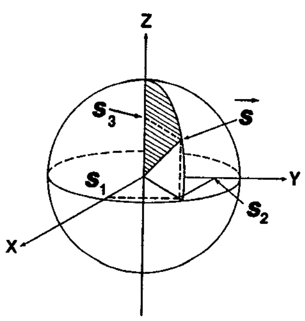
$$= \mathcal{S}_{n0}\ s_0 + |\vec{\mathcal{S}}_n|\ |\vec{a}|\ \cos\ 2\Psi_n \quad (12b)$$

$$= \mathcal{S}_{n0}\ s_0 + \mathcal{S}_{n0} s_0\ \cos\ 2\Psi_n \quad (12c)$$

$$= 2\ \mathcal{S}_{n0} s_0\ (\cos\ \Psi_n)^2 \quad (12d)$$

Stokes vector $\underline{\mathcal{S}}$ is shown in a Poincare sphere in Fig. 2.

FIGURE 2. POINCARE SPHERE AND STOKES VECTOR *S*



To apply these results directly to the six-port reflection network analyzer, it is only necessary to replace the conventional scattering matrix in (1) with another one containing the same information and easily derivable from the original matrix. In particular, we set

$$\underline{x}_\alpha = \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}, \text{ and } \underline{y}_\alpha = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} \quad (13)$$

Then

$$\begin{bmatrix} \underline{x}_\alpha \\ \underline{a}_\beta \end{bmatrix} = \begin{bmatrix} N_{\alpha\alpha} & N_{\alpha\beta} \\ 0 & 1_{\beta\beta} \end{bmatrix} \begin{bmatrix} \underline{a}_\alpha \\ \underline{a}_\beta \end{bmatrix} \quad (14a)$$

$$\begin{bmatrix} \underline{y}_\alpha \\ \underline{b}_\beta \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{bmatrix} \begin{bmatrix} \underline{a}_\alpha \\ \underline{a}_\beta \end{bmatrix} \quad (14b)$$

where, for example, from (1) we find

$$[N_{\alpha\alpha} \quad N_{\alpha\beta}] = \left[\begin{array}{cc|cccc} S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \quad (15)$$

The desired non-conventional "scattering matrix" equation is therefore

$$\begin{bmatrix} \underline{y}_\alpha \\ \underline{b}_\beta \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{bmatrix} \begin{bmatrix} N_{\alpha\alpha} & N_{\alpha\beta} \\ 0 & 1_{\beta\beta} \end{bmatrix}^{-1} \begin{bmatrix} \underline{x}_\alpha \\ \underline{a}_\beta \end{bmatrix} \quad (16)$$

The required inverse

$$\begin{bmatrix} N_{\alpha\alpha} & N_{\alpha\beta} \\ 0 & 1_{\beta\beta} \end{bmatrix}^{-1} = \begin{bmatrix} N_{\alpha\alpha}^{-1} & -N_{\alpha\alpha}^{-1} N_{\alpha\beta} \\ 0 & 1_{\beta\beta} \end{bmatrix} \quad (17)$$

evidently exists provided only $S_{21} \neq 0$. Employing (16) and (2) to redefine the column 4-vectors in (3b) [$\underline{x}_\alpha$ replacing $\underline{a}_\alpha$], we obtain

$$\underline{b}_\beta = \underline{A}'_\beta\ \ b_2 + \underline{B}'_\beta\ \ a_2 \quad (18)$$

We now proceed as before, paralleling equations (4) through (12), to express P_n as the product of Stokes vectors. In particular, paralleling (5),

$$\hat{\underline{\Gamma}} = \begin{bmatrix} |b_2|^2 \\ b_2a_2^* \\ a_2b_2^* \\ |a_2|^2 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & j & 0 \\ 0 & 1 & -j & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} |b_2|^2 \begin{bmatrix} 1 + |\Gamma|^2 \\ 2\mathrm{Re}\ \ \Gamma^* \\ 2\mathrm{Im}\ \ \Gamma^* \\ 1 - |\Gamma|^2 \end{bmatrix} \quad (19)$$

In this equation we have inserted the reflection coefficient at port 2, $a_2 = \Gamma\ b_2$, dropping the subscript 2 on Γ for convenience when no confusion with the matrix Γ defined in (22b) arises. We will also find it convenient to denote the Stokes vector

$$\underline{\Gamma} = \begin{bmatrix} 1 + |\Gamma|^2 \\ 2\ \mathrm{Re}\ \Gamma \\ 2\ \mathrm{Im}\ \Gamma \\ 1 - |\Gamma|^2 \end{bmatrix} \quad (20)$$

keeping separate the amplitude factor $|b_2|^2$.

CALIBRATION

We assume that five terminations [8] with known reflection coefficients which meet a restriction to be given shortly are available. Power readings from the four power meters corresponding to each of the five known terminations are noted, that is, we have five sets of readings

$$\underline{\tilde{P}}^{(n)} = [P_3^{(n)}\ P_4^{(n)}\ P_5^{(n)}\ P_6^{(n)}],\ n = 1,2,\ \ldots,\ 5 \quad (21)$$

corresponding to five Stokes vectors $\underline{\Gamma}^{(n)}$ as in (20). Defining the square 4 x 4 matrices

$$P = [\underline{P}^{(1)}\ \underline{P}^{(2)}\ \underline{P}^{(3)}\ \underline{P}^{(4)}] \quad (22a)$$

$$\Gamma = [\underline{\Gamma}^{(1)}\ \ \underline{\Gamma}^{(2)}\ \ \underline{\Gamma}^{(3)}\ \ \underline{\Gamma}^{(4)}] \quad (22b)$$

$$B = [\,|b_2^{(n)}|^2\ \delta_{nm}],\ n,\ m = 1,\ 2,\ 3,\ 4. \quad (22c)$$

we may write, from equation (18), (19) and the

parallel of (7)

$$P = \tfrac{1}{2}C'\ T\ \Gamma\ B, \quad (23a)$$

$$\tfrac{1}{2}C'\ T = P\ B^{-1}\ \Gamma^{-1} \quad (23b)$$

wherein the diagonal elements of B are as yet undetermined. The fifth set of measurements now determines the elements of B from the four linear equations

$$\underline{P}^{(5)} = \tfrac{1}{2}C'\ T\ \underline{\Gamma}^{(5)}|b_2^{(5)}|^2 = P\ B^{-1}\Gamma^{-1}\ \underline{\Gamma}^{(5)}|b_2^{(5)}|^2 \quad (24)$$

in terms of the scale factor $|b_2^{(5)}|^2$ which we may set at an arbitrary positive magnitude. With B found, $\frac{1}{2}$ C' T can be computed from (23), and the calibration is complete. The arbitrary scale factor introduced into B, and inversely into C' T, cancels from (23a).

The reader may have remarked the absence of the carat over the power quantities P. The carat was employed to distinguish incident from total power. In calibration (as opposed to computation from given circuit elements) we may lump any mismatch of the detector into the six-port. Then power absorbed by a meter can be set equal to the incident power.

We now return to the promised restriction on the choice of the five $\Gamma^{(n)}$. Clearly, the inverse of Γ (22b) must exist by (23) and (24). Thus we must ensure that

$$\det \Gamma \neq 0, \quad (25)$$

i.e., the four $\underline{\Gamma}^{(n)}$ must be linearly independent. If the end points of the corresponding four 3-space vectors $\vec{\Gamma}^{(n)}$ are coplanar, then the four $\underline{\Gamma}^{(n)}$are linearly dependent [9]. We may represent reflection coefficients by normalized $\underline{\Gamma}^{(n)}$ on the Poincare sphere. If they are linearly dependent, the plane through their endpoints determines a circle on this sphere. By stereographic projection, the same values of $\Gamma^{(n)}$ lie on a circle in the complex reflection coefficient plane [9,12]. It is therefore clear that any four reactive terminations, which lie on a circle in Γ plane, will always lead to linearly dependent $\Gamma^{(n)}$, and (as is now well known) are therefore insufficient for purposes of calibration.

MEASUREMENT

In performing one measurement, we record the readings of the four power meters in a column vector $\underline{P}$. Using the result of the calibration (23) to define a matrix $\mathcal{S} = 2$ C' T

$$\underline{P} - \mathcal{S}\,\underline{\Gamma}\ ,\quad \underline{\Gamma} = \mathcal{S}^{-1}\ \underline{P} \quad (26)$$

It will be remembered that $\mathcal{S}$ contains the arbitrarily set scale factor $|b_2{}^{(5)}|^2$. We may always choose to "normalize" the components of $\underline{\Gamma}$. In practice, this is best accomplished by leaving the matrix $\mathcal{S}$ fixed, but dividing the elements of both $\underline{P}$ and $\underline{\Gamma}$ as found from (26) by $\frac{1}{2}$ $(\Gamma_o + \Gamma_3)$. We denote such normalized values by $°\underline{P}$ and $°\underline{\Gamma}$. The space vector part of $°\underline{\Gamma}$,

$$°\vec{\Gamma} = °\Gamma_1\hat{x} + °\Gamma_2\hat{y} + °\Gamma_3\hat{z}\ , \quad (27)$$

then defines a point on a Poincare sphere of radius $1 + |\Gamma|^2 = °\Gamma_o$ corresponding to the measured reflection coefficient. As already mentioned, points on the sphere corresponding directly to points on the complex reflection coefficient plane explicitly in terms of the normalized $°\underline{\Gamma}$, by

$$\mathrm{Re}\ \Gamma = \tfrac{1}{2}\ °\Gamma_1,\ -\ \mathrm{Im}\ \Gamma = \tfrac{1}{2}\ °\Gamma_2 \quad (28)$$

The measurement can readily be given a geometrical interpretation on the Poincare sphere (or by stereographic projection) on the complex plane by means of equation (12). We recall that the rows of S are Stokes vectors (7) and (8). The directions of the space portions of these rows, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{S}_6$, determine axes in space. Circular cones with vertex angles 2 Ψn about these axes,

$$\Psi_n = \arccos \sqrt{\frac{°P_n}{\mathcal{S}_{no}\ °\Gamma_o}}, \quad (29)$$

c.f. Eq. (12) cut a Poincare sphere in four circles. One such construction is shown in Fig. 3.

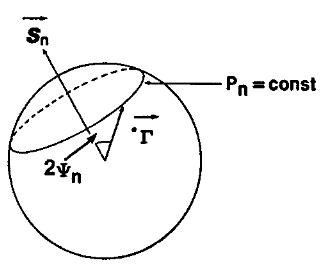
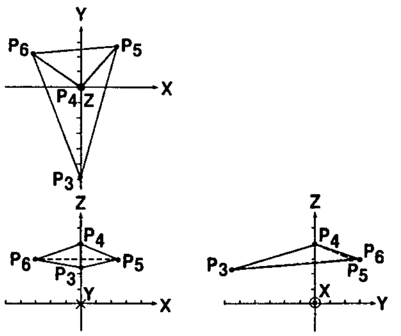


FIGURE 3. CONSTANT POWER CIRCLE ON POINCARE SPHERE

These four circles will intersect at a point on the sphere of appropriate radius $°\Gamma_0$ corresponding to the measured Γ. It is now evident that, in order to yield a well defined intersection, the centers of these circles should be widely spaced. Stokes vectors corresponding to the six-port circuit employed by Engen [3] are shown in Fig. 4. It will be seen that they are very well spaced.

FIGURE 4. ENGEN'S CIRCUIT: STOKES VECTOR REPRESENTATION



SIMPLE SIX-PORT CRITERIA

Based on the elements of the scattering matrix (1) we may develop two simple (sufficient) criteria which can serve to make a given six-port unsuitable as an analyzer.

Consider utilization of a six-port with matched power detectors, $Q_n = 0$. Then, in accordance with (3a), the matrix C is determined by the elements of $S_{\beta\alpha}$. Evidently, if C is singular, then the junction is unsuitable. It may be shown that C is singular when:

i) Any two rows of $S_{\beta\alpha}$ are linearly dependent

ii) The two columns of $S_{\beta\alpha}$ are linearly dependent

When these criteria are applied to three well-known symmetrical lossless reciprocal six-ports:

1. Purcell's junction
2. The Turnstile junction
3. The six-armed Star junction

it follows from symmetry analysis that none of these junctions is suitable as a six-port analyzer. In the instance of the six-armed star junction this conclusion is in agreement with Riblet [10].

ACKNOWLEDGEMENT

It is a pleasure to acknowledge stimulating discussions with Dr. L. Susman.

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